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ON OPTIMAL PROBLEMS OF THE THEORY OF ELASTICITY WITH UNKNOWN BOUNDARIES*

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In optimal-design problems in the theory of elasiticty when the shape of the boundary is sought /1—3/, the domain varies and in the long run is subject to determination, unlike design problems when the elastic moduli of the material are unknown /4, 5/. The solution of such problems is often irregular in nature /2/. In this connection, the need arises for a classification of a suitable set of allowable domains that can be given by using two parameters. For this set of domains a variational concept is presented and a theorem is proved on the existence of variations of the displacements of an elastic structure.

1. The class of domains under consideration. Let R^n be an n-dimensional Euclidean space of vectors $\mathbf{x}$ ($n = 2$ or 3) in which a Cartesian coordinate system is defined by the directions $\mathbf{e}_i$ such that $\mathbf{x} = x_i\mathbf{e}_i$. Here and henceforth, the Latin subscripts take the values $1, \ldots, n$; summation from 1 to n is assumed over the repeated subscripts in the products.

Definition 1. The set /6/

$$\Gamma = \{\mathbf{y} \in R^n \mid \mathbf{y} = \mathbf{y}(\tau^\circ),\ \tau^\circ \in T\}$$

$$T = \{\tau^\circ = (\tau^2, \ldots, \tau^n) \mid 0 < \tau^\alpha < 1,\ 0 < \sum_{\alpha=2}^{n} \tau^\alpha < 1\}$$

$$\mathbf{y}_k(\tau^\circ) \in C^m(\bar{T}),\quad m \geqslant 1$$

is called a differentiable $(n-1)$-dimensional cell, when C^m is a space of m times differentiable functions, the vectors $\mathbf{r}_\alpha = \partial\mathbf{y} \setminus \partial\tau^\alpha$ are linearly independent for $\forall\tau^\circ \in \bar{T}$, i.e., form a covariant moving basis of the coordinate system τ^α /7/, the mapping $\mathbf{y}_k(\tau^\circ)$ is one-to-one in $\bar{T}$. Here and henceforth, the Greek indices take the values $2, \ldots, n$, and summation from 2 to n is assumed in the repeated super- and subscripts in the products.

We determine the normal direction $\mathbf{r}_1 = \mathbf{r}_1(\mathbf{y})$ at each point $\mathbf{y} \in \Gamma$, where we select its direction such that

$$Y = Y(\tau^\circ) = \det \| \mathbf{r}_1\mathbf{r}_2 \ldots \mathbf{r}_n \| > 0 \tag{1.1}$$

The area element of the surface Γ is determined by the formula /8/

$$d\Gamma = Y(\tau^\circ)\, d\tau^2 \ldots d\tau^n = Y(\tau^\circ)\, d\tau^\circ$$

We introduce curvilinear coordinates /7/

$$\mathbf{x}(\tau) = \mathbf{y}(\tau^\circ) + \tau^1\mathbf{r}_1(\mathbf{y}(\tau^\circ)),\quad \tau = (\tau^1, \ldots, \tau^n) \tag{1.2}$$

in the neighbourhood of Γ and we calculate the covariant vectors

$$\mathbf{R}_\alpha(\tau) = \partial\mathbf{x}/\partial\tau^\alpha = \mathbf{r}_\alpha - \tau^1\mathbf{r}_\alpha\cdot\mathbf{t},\quad \mathbf{R}_1(\mathbf{y}) = \mathbf{r}_1(\mathbf{y}) \tag{1.3}$$

where $\mathbf{t} = \mathbf{t}(\mathbf{y}(\tau^\circ)) = -\nabla^\circ\mathbf{r}_1$ is the curvature tensor of the surface Γ, $\nabla^\circ = \mathbf{r}^\alpha\,\partial/\partial\tau^\alpha$ is an $(n-1)$-dimensional Hamilton operator, and $\mathbf{r}^\alpha$ is the contravariant basis of the coordinate system τ^α. The tensor $\mathbf{t}$ and the direction $\mathbf{r}_1$ depend only on the cells Γ but not on the selection of the coordinates τ°.

Let $X(\tau) = \det \| \mathbf{R}_1\mathbf{R}_2 \ldots \mathbf{R}_n \|$ be the Jacobi matrix of the coordinate transformation $\mathbf{x}(\tau)$. It follows from (1.3) that

$$X(\tau) = Y(\tau^\circ)\,[1 + I_1(\mathbf{t})(-\tau^1) + \ldots + I_{n-1}(\mathbf{t})(-\tau^1)^{n-1}]$$

*Prikl.Matem.Mekhan., 50, 2, 231-236, 1986 where $I_k(\mathbf{t})$ are invariants of the curvature tensor $\mathbf{t}$ /9/. The quantity $I_1(\mathbf{t})/(n-1)$ is called the mean while $I_{n-1}(\mathbf{t})$ is the Gaussian curvature /9/. The condition (1.1) is satisfied for Y, therefore, $\exists\lambda > 0$ such that

$$X(\tau^1, \tau^\circ) > 0, \ \forall\tau^\circ \in \bar{T}, \ \ |\tau^1| \leqslant \lambda \tag{1.4}$$

The volume element in the neighbourhood of Γ is defined by the formula /8/

$$dx = dx_1 \ldots dx_n = X(\tau)\, d\tau^1 \ldots d\tau^n = X(\tau)\, d\tau$$

Definition 2. We call the function $r(\mathbf{y}, \eta)$, where $\mathbf{y} \in \Gamma$ and η is a parameter, a generating boundary function (GBF) if $\exists\eta_0 > 0$ such that for all $0 < \eta < \eta_0$

$$r(\mathbf{y}, \eta), \ \ \partial r(\mathbf{y}, \eta)/\partial\eta, \ \ \partial^2 r(\mathbf{y}, \eta)/\partial\eta^2 \in \dot{C}^m(T) \tag{1.5}$$
$$r(\mathbf{y}, 0) \equiv 0, \ \ |r(\mathbf{y}, \eta)| \leqslant \lambda$$

where $\mathbf{y} = \mathbf{y}(\tau^\circ)$, and λ is the quantity in (1.4).

The GBF enables us to obtain $(n-1)$-dimensional cells

$$\Gamma(\eta) = \{\mathbf{y}(\tau^\circ, \eta) \mid \mathbf{y}(\tau^\circ, \eta) = \mathbf{y} + r(\mathbf{y}, \eta)\, \mathbf{r}_1(\mathbf{y}), \ \mathbf{y} = \mathbf{y}(\tau^\circ), \ \ \tau^\circ \in \bar{T}\}$$

There follows from the conditions (1.5) that $\Gamma(0) = \Gamma$.

Definition 3. We call the functions $\delta r(\mathbf{y}) = \partial r(\mathbf{y}, 0)/\partial\eta$ and $\delta^2 r(\mathbf{y}) = \partial^2 r(\mathbf{y}, 0)/\partial\eta^2$, respectively, the first and second variations of the $(n-1)$-dimensional cell Γ.

Let us obtain an expression for the variation in the direction of the normal $\mathbf{r}_1$. To do this, we find the vector of the covariant basis

$$\mathbf{r}_\alpha(\tau^\circ, \eta) = \mathbf{r}_\alpha - r(\mathbf{y}, \eta)\, \mathbf{r}_\alpha \cdot \mathbf{t} + \frac{\partial r(\mathbf{y}, \eta)}{\partial\tau^\alpha}\, \mathbf{r}_1(\mathbf{y})$$

The direction $\mathbf{r}_1(\mathbf{y}, \eta)$ is defined by the equations $|\mathbf{r}_1(\mathbf{y}, \eta)| = 1$, $\mathbf{r}_1(\mathbf{y}, \eta) \cdot \mathbf{r}_\alpha(\tau^\circ, \eta) = 0$, from which we find

$$\mathbf{r}_1(\mathbf{y}, \eta) = \mathbf{r}_1 - \nabla^\circ r(\mathbf{y}, \eta) - \frac{1}{2} |\nabla^\circ r(\mathbf{y}, \eta)|^2\, \mathbf{r}_1 - r(\mathbf{y}, \eta)\, \mathbf{t} \cdot \nabla^\circ r(\mathbf{y}, \eta) + o(r^2(\mathbf{y}, \eta)) \tag{1.6}$$

Definition 4. The vectors $\delta\mathbf{r}_1(\mathbf{y}) = \partial\mathbf{r}_1(\mathbf{y}, 0)/\partial\eta$, $\delta^2\mathbf{r}_1(\mathbf{y}) = \partial^2\mathbf{r}_1(\mathbf{y}, 0)/\partial\eta^2$ will respectively be called the first and second variations of the direction $\mathbf{r}_1$.

We find from (1.6)

$$\delta\mathbf{r}_1(\mathbf{y}) = -\nabla^\circ\delta r(\mathbf{y}), \ \ \delta^2\mathbf{r}_1(\mathbf{y}) = -\nabla^\circ\delta^2 r(\mathbf{y}) -$$
$$|\nabla^\circ\delta r(\mathbf{y})|^2\, \mathbf{r}_1 - 2\delta r(\mathbf{y})\, \mathbf{t} \cdot \nabla^\circ\delta r(\mathbf{y})$$

Definition 5. We shall say that the domain $\Omega \subset R^n$ is referred to the class $D^m(\lambda)$, $0 < \lambda < 1$ if:

1) Ω is a bounded domain with piecewise-smooth boundary Γ, which dissociates into a finite number $\leqslant \lambda^{-1}$ of non-intersecting /6/ m-times differentiable $(n-1)$-dimensional cells;

2) The curvilinear coordinates (1.2), for which $X(\tau) > 0$, can be introduced in the neighbourhood $|\tau^1| < \lambda$ or each cell;

3) For $\forall\mathbf{y} \in \Gamma$ and any $0 < \delta < \lambda$ the domains Ω, $\Omega \bigcup\limits_{\mathbf{y}\in\Gamma} U(\mathbf{y}, \delta)$, $\Omega \setminus \overline{\bigcup\limits_{\mathbf{y}\in\Gamma} U(\mathbf{y}, \delta)}$ have identical connectness and

$$\operatorname{mes} \Omega \cap U(\mathbf{y}, \delta) \geqslant \lambda \operatorname{mes} U(\mathbf{y}, \delta)$$
$$\operatorname{mes} U(\mathbf{y}, \delta) \setminus \bar{\Omega} \geqslant \lambda \operatorname{mes} U(\mathbf{y}, \delta)$$
$$U(\mathbf{y}, \delta) = \{\mathbf{x} \in R^n \mid |\mathbf{x} - \mathbf{y}| < \delta\}$$

We shall consider the direction of the normal $\mathbf{r}_1(\mathbf{y})$ to be external at each regular point.

2. Elastic constructions. *Definition 6.* We shall call $\Omega^\circ \subset R^n$ a projection domain if it is of the class $D^m(\lambda)$ and its boundary consists of the three non-intersecting sets Γ_u°, Γ_F° and Γ_0° ($\Gamma^\circ = \bar{\Gamma}_u^\circ \cup \bar{\Gamma}_F^\circ \cup \bar{\Gamma}_0^\circ$), on which the following vectors are given: the displacements

$$\mathbf{u}(\mathbf{y}) = u_i(\mathbf{y})\, \mathbf{e}_i, \ \ \mathbf{u} \in V(\Omega^\circ) = \{\mathbf{u} \mid u_i \in W_2^1(\Omega^\circ), u_i(\mathbf{y}) = 0, \ \ \ \mathbf{y} \in \Gamma_u^\circ\}$$

and the surface loads

$$\mathbf{F}(\mathbf{y}) = F_i(\mathbf{y})\, \mathbf{e}_i, \ \ F_i(\mathbf{y}) \in L_2(\Gamma_F^\circ) \tag{2.1}$$

Here W_2^1 is the space of Sobolev functions that are square summable together with the first generalized derivative, and L_2 is the square-summable space of functions /10/.

Definition 7. We shall call $\Omega \subset \Omega^\circ$ an allowable domain if Ω is of the class $D^m(\lambda)$ and its boundary satisfies the conditions

$$\Gamma_u = \Gamma \cap \Gamma_u^\circ, \ \ \Gamma_F = \Gamma \cap \Gamma_F^\circ = \Gamma_F^\circ$$

We denote the set of allowable domains by $O^m(\lambda)$. The inclusions $O^m(\lambda_2) \subset O^m(\lambda_1)$, $\lambda_2 \geqslant \lambda_1$ are obvious.

We assume the domain $\Omega \in O^m(\lambda)$ to be filled with an elastic homogeneous material characterized by the elastic constants tensor

$$\mathbf{a} = a_{ijkl}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l, \quad a_{ijkl} = a_{klij} = a_{jikl}$$

We call the domain Ω filled with the elastic material an elastic construction.

A load (2.1) acts on the surface Γ_F whereupon the elastic construction performs the displacement

$$\mathbf{u}(\mathbf{x}) \in V(\Omega) = \{\mathbf{u} \mid u_i \in W_2^1(\Omega), \ u_i(\mathbf{y}) = 0, \ \mathbf{y} \in \Gamma_u\}$$

which can be determined from an integral identity /11/

$$\int_\Omega \pi(\mathbf{u}, \mathbf{v})\,dx - \int_{\Gamma_F} F_i v_i\, d\Gamma = 0, \quad \forall \mathbf{v} \in V(\Omega) \tag{2.2}$$

$$\pi(\mathbf{u}, \mathbf{v}) = \boldsymbol{\varepsilon}(\mathbf{u}) \cdot\cdot\, \mathbf{a} \cdot\cdot\, \boldsymbol{\varepsilon}(\mathbf{v}) = a_{ijkl}\varepsilon_{ij}(\mathbf{u})\,\varepsilon_{kl}(\mathbf{v})$$

$$\boldsymbol{\varepsilon} = \varepsilon_{ij}\mathbf{e}_i\mathbf{e}_j, \quad \varepsilon_{ij}(\mathbf{v}) = (v_{i,j} + v_{j,i})/2 \in L_2(\Omega)$$

($\pi(\mathbf{u}, \mathbf{v})$ is the bilinear form in the strain tensors $\boldsymbol{\varepsilon}(\mathbf{u})$, $\boldsymbol{\varepsilon}(\mathbf{v})$).

We denote the norms of the functions introduced, respectively, by /10/

$$\|\mathbf{F}\|_{2,\Gamma_F} = \Big(\int_{\Gamma_F} |\mathbf{F}|^2\, d\Gamma\Big)^{1/2}, \quad \|\boldsymbol{\varepsilon}\|_{2,\Omega} = \Big(\int_\Omega \varepsilon_{ij}\varepsilon_{ij}\, dx\Big)^{1/2}$$

$$\|\mathbf{u}\|_{2,\Omega}^{(1)} = \Big[\int_\Omega (|\mathbf{u}|^2 + u_{i,j}u_{i,j})\,dx\Big]^{1/2}$$

Let either $\operatorname{mes}\Gamma_u \neq 0$ or $\operatorname{mes}\Gamma_u = 0$ and

$$\int_{\Gamma_F} F_i w_i\, d\Gamma = 0, \quad \forall \mathbf{w} \in V_0(\Omega) \subset V(\Omega)$$

where $V_0(\Omega)$ is the set of all "rigid" displacements /11/. In these cases the existence theorem for a solution of the integral identity (2.2) holds /11/, where

$$\|\mathbf{u}\|_{2,\Omega}^{(1)} \leqslant c\,\|\mathbf{F}\|_{2,\Gamma_F} \tag{2.3}$$

3. The existence of variations of the displacements. The domain Ω remains fixed in problems of control of the elastic characteristics of the material, and the question of the existence of variations of the displacements is connected directly with the "smoothness" of the coefficients of the equations and is trivial. If the domain changes, then it should be clarified when the variations of the displacements that are dependent on the variations of the domain will exist in that same space as the displacement itself.

Let $\Omega^* \in O^m(3\lambda)$, $0 < 3\lambda < 1$.

Definition 8. We call the function $r(\mathbf{y}, \eta)$, $\mathbf{y} \in \Gamma^*$, $0 \leqslant \eta \leqslant \eta_0$, $\eta_0 > 0$ an allowable GBF (AGBF) if the function $r(\mathbf{y}, \eta)$ is a GBF in each $(n-1)$-dimensional cell Γ^*, where

$$r(\mathbf{y}, \eta) = \begin{cases} 0, & \mathbf{y} \in \bar{\Gamma}_1^* = \bar{\Gamma}_F^* \cup \bar{\Gamma}_u^* \\ \leqslant 0, & \mathbf{y} \in \bar{\Gamma}_2^* = (\overline{\Gamma^\circ \cap \Gamma^*}) \setminus \Gamma_1^* \\ \leqslant 0 \text{ or } \geqslant 0, & \mathbf{y} \in \Gamma_3^* = \Gamma^* \setminus (\bar{\Gamma}_1^* \cup \bar{\Gamma}_2^*) \end{cases}$$

and the domain $\Omega(\eta)$ bounded by $\Gamma(\eta)$ belongs to $O^m(\lambda)$.

Let $\mathbf{u}(\mathbf{x}, \eta) \in V(\Omega(\eta))$ denote the solution of the integral identity (2.2). It is known /10/ that if $\Omega(\eta) \in O^m(\lambda)$, then for $\forall \mathbf{v} \in V(\Omega(\eta))$

$$\exists \mathbf{v}^\circ \in V(\Omega^\circ), \quad \mathbf{v}^\circ(\mathbf{x}) = \mathbf{v}(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega(\eta) \tag{3.1}$$

The function $\mathbf{v}^\circ$ is called the continuation of $\mathbf{v}$ in the domain Ω°. We shall henceforth always consider such a continuation satisfied and we retain the notation $\mathbf{v}$ for $\mathbf{v}^\circ$

Theorem 1. Let $0 < 3\lambda < 1$, $m \geqslant 3$, $\Omega^* \in O^m(3\lambda)$, and let $r(\mathbf{y}, \eta)$ be a AGBF for $0 \leqslant \eta \leqslant \eta_0$. Then $\exists \partial \mathbf{u}(\mathbf{x}, \eta)/\partial\eta \in V(\Omega(\eta))$ for any $0 \leqslant \eta \leqslant \eta_0$.

Proof. We take $0 < \eta < \eta_0$ and an $\Delta\eta$ such that $0 \leqslant \eta \pm \Delta\eta \leqslant \eta_0$. We use the notation $\eta^+ = \eta + \Delta\eta$

$$\Omega = \Omega(\eta), \quad \Gamma = \Gamma(\eta), \quad \Omega^+ = \Omega(\eta^+), \quad \Gamma^+ = \Gamma(\eta^+)$$

We denote the solutions of the integral identity for the domains Ω and Ω^+ by $\mathbf{u}(\mathbf{x}, \eta)$ and $\mathbf{u}^+(\mathbf{x}, \eta^+)$, respectively.

Without loss of generality, it can be assumed that the function $r(\mathbf{y}, \eta)$ is given in just one $(n-1)$-dimensional cell of the set Γ_3^* where, to be specific, we put $r(\mathbf{y}, \eta) \geqslant 0$.

Let us consider the integral identity

$$\int_{\Omega^+} \pi(\mathbf{u}^+, \mathbf{v})\, dx - \int_{\Omega} \pi(\mathbf{u}, \mathbf{v})\, dx = 0, \quad \forall \mathbf{v} \in V(\Omega^+)$$

which is satisfied by virtue of (2.2), (3.1) and we represent it in the form

$$\int_{\Omega} \pi(\mathbf{u}^+ - \mathbf{u}, \mathbf{v})\, dx = - \int_{\Omega^+ \setminus \overline{\Omega}} \pi(\mathbf{u}^+, \mathbf{v})\, dx \tag{3.2}$$

We note that the function $\mathbf{u}^+$ is differentiable $m-1$ times /12/ in $\Omega^+ \setminus \overline{\Omega}$. The formula

$$\pi(\mathbf{w}, \mathbf{v}) = \nabla \cdot [\sigma(\mathbf{w}) \cdot \mathbf{v}] - [\nabla \cdot \sigma(\mathbf{w})] \cdot \mathbf{v} \tag{3.3}$$

holds for the twice differentiable function $\mathbf{w}$ where $\sigma = \mathbf{a} \cdot\cdot\, \varepsilon$ is the stress tensor.

Using (3.3) on the right side of the integral identity (3.2) and also taking into account that $\nabla \cdot \sigma(\mathbf{u}^+) = \mathbf{0}$, $\mathbf{x} \in \Omega^+ \setminus \overline{\Omega}$ and

$$\mathbf{r}_1 \cdot \sigma(\mathbf{u}^+) = 0, \quad \forall \mathbf{y}^+ = \mathbf{y}^* + r(\mathbf{y}^*, \eta^+)\, \mathbf{r}_1(\mathbf{y}^*) \tag{3.4}$$

we obtain

$$\int_{\Omega} \pi(\mathbf{u}^+ - \mathbf{u}, \mathbf{v})\, dx = \int_{\Gamma} \mathbf{r}_1 \cdot \sigma(\mathbf{u}^+) \cdot \mathbf{v}\, d\Gamma, \quad \forall \mathbf{v} \in V(\Omega) \tag{3.5}$$

It follows from this integral identity and (2.3) that

$$\| \mathbf{u}^+ - \mathbf{u} \|_{2, \Omega}^{(1)} \leqslant c \, \| \mathbf{r}_1 \cdot \sigma(\mathbf{u}^+) \|_{2, \Gamma} \tag{3.6}$$

where the constant c can be selected to be independent of η.

We take account of (1.6) and expand the function in a Taylor series

$$\begin{aligned} &\mathbf{r}_1(\mathbf{y}^+) \cdot \sigma(\mathbf{u}^+(\mathbf{y}^+, \eta^+)) = \mathbf{r}_1(\mathbf{y}) \cdot \sigma(\mathbf{u}^+(\mathbf{y}, \eta^+)) + \\ &\quad \mathbf{r}_1(\mathbf{y}) \cdot \partial\sigma(\mathbf{u}^+(\mathbf{y}, \eta^+))/\partial\tau^1 \partial r/\partial\eta \Delta\eta - \nabla^\circ (\partial r/\partial\eta) \cdot \\ &\cdot \sigma(\mathbf{u}^+(\mathbf{y}, \eta^+))\, \Delta\eta + o(\Delta\eta) \\ &\forall \mathbf{y} \in \Gamma, \quad \partial r/\partial\eta = \partial r(\mathbf{y}^*, \eta)/\partial\eta \end{aligned} \tag{3.7}$$

The left side in (3.7) equals zero because of (3.4), consequently, it follows from (3.6) that

$$\| \mathbf{u}^+ - \mathbf{u} \|_{2, \Omega}^{(1)} \to 0 \quad \text{as} \quad \Delta\eta \to 0 \tag{3.8}$$

We denote by $\mathbf{w} \in V(\Omega)$ the solution of the integral identity

$$\int_{\Omega} \pi(\mathbf{w}, \mathbf{v})\, dx = - \int_{\Gamma} f \cdot \mathbf{v}\, d\Gamma, \quad \forall \mathbf{v} \in V(\Omega) \tag{3.9}$$

$$\mathbf{f} = \mathbf{r}_1 \cdot \frac{\partial\sigma(\mathbf{u})}{\partial\tau^1} \frac{\partial r}{\partial\eta} - \nabla^\circ \frac{\partial r}{\partial\eta} \cdot \sigma(\mathbf{u})$$

The function $\mathbf{w}$ is $m-1$ times differentiable near Γ, consequently

$$\mathbf{r}_1 \cdot \sigma(\mathbf{w}) = -\mathbf{f}, \quad \forall \mathbf{y} \in \Gamma$$

We subtract the integral identity (3.5) divided by $\Delta\eta$ from (3.9). We then obtain the relationship

$$\int_{\Omega} \pi\left(\mathbf{w} - \frac{\mathbf{u}^+ - \mathbf{u}}{\Delta\eta}, \mathbf{v}\right) dx = - \int_{\Gamma} \left[\mathbf{f} + \frac{\mathbf{r}_1 \cdot \sigma(\mathbf{u}^+)}{\Delta\eta}\right] \cdot \mathbf{v}\, d\Gamma, \quad \forall \mathbf{v} \in V(\Omega) \tag{3.10}$$

from which and from (2.3) it follows that

$$\| \mathbf{w} - (\mathbf{u}^+ - \mathbf{u})/\Delta\eta \|_{2, \Omega}^{(1)} \leqslant c \, \| \mathbf{f} + \mathbf{r}_1 \cdot \sigma(u^+)/\Delta\eta \|_{2, \Gamma} \tag{3.11}$$

It follows from (3.8) and (3.5) that $\mathbf{u}^+ \to \mathbf{u}$ near Γ and point-by-point together with the first derivatives as $\Delta\eta \to 0$ Taking account of (3.7), we find from (3.11)

$$\| \mathbf{w} - (\mathbf{u}^+ - \mathbf{u})/\Delta\eta \|_{2, \Omega}^{(1)} \leqslant c \, \| o(\Delta\eta)/\Delta\eta \|_{2, \Gamma} \tag{3.12}$$

from which the theorem follows.

Theorem 2. Let $m \geqslant 4$ and the conditions of Theorem 1 be satisfied. Then

$$\partial^2 \mathbf{u}(\mathbf{x}, \eta)/\partial\eta^2 \in V(\Omega(\eta))$$

The proof of this theorem is exactly the same as the preceding. The difference is that now a finite difference $(\mathbf{u}^+ - 2\mathbf{u} + \mathbf{u}^-)/\Delta\eta^2$ must be constructed, the expansion (1.6) must be used, and considerable smoothness of the $(n-1)$-dimensional cell and the AGBF $r(\mathbf{y}^*, \eta)$ must be required.

Remark 1°. The quantities $\delta\mathbf{u} = \partial\mathbf{u}(\mathbf{x}, 0)/\partial\eta$ and $\delta^2\mathbf{u} = \partial^2\mathbf{u}(\mathbf{x}, 0)/\partial\eta^2$ are, respectively, called the first and second variations of the displacement $\mathbf{u}$. It follows from Theorems 1 and 2 that $\delta\mathbf{u},\ \delta^2\mathbf{u} \in V(\Omega^*)$.

Remark 2°. If $r(\mathbf{y}^*, \eta) < 0$ the proof of Theorem 1 will differ somewhat from that presented. In particular, the differentiability of $\mathbf{u}$ and $\mathbf{w}$ should be taken into account in $\Omega \setminus \overline{\Omega}^*$ and Ω should be replaced by Ω^+ in the integral identities (3.2), (3.5), (3.10), the estimates (3.6), (3.11), (3.12), and in the limit (3.8).

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ON CORRECT FORMULATIONS OF LEKHNITSKII PROBLEMS*

N.KH. ARUTYUNYAN, A.B. MOVCHAN, and S.A. NAZAROV

The deformation of an elastic half-space with a cylindrical cavity under its own weight is considered. Since the solution of the problem increases at infinity, the question arises of its uniqueness and of the correct formulation of the problem itself. It is shown that two such formulations exist that yield unique solutions (that differ only to the accuracy of rigid displacements). The former corresponds to a decrease in the displacement u_j in a layer abutting on the half-space boundary, and the latter to a decrease in the stress tensor components σ_{jk}, $j, k = 1, 2$. The solutions corresponding to these formulations are distinct. They can be obtained by a passage to the limit as $D \to \infty$ from solutions of problems on the deformation of a semicylinder of diameter D with a coaxial cylindrical cavity; in the first case the side surface of the cylinder is considered rigidly clamped, and in the second stress-free.

The results are generalized to the case of non-symmetric paraboloidal cavities and elastic inclusions. Formulations are discussed of problems in which the force of gravity depends on the distance to the half-space boundary.

1. The boundary value problem and its particular solutions. Let g be a domain

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